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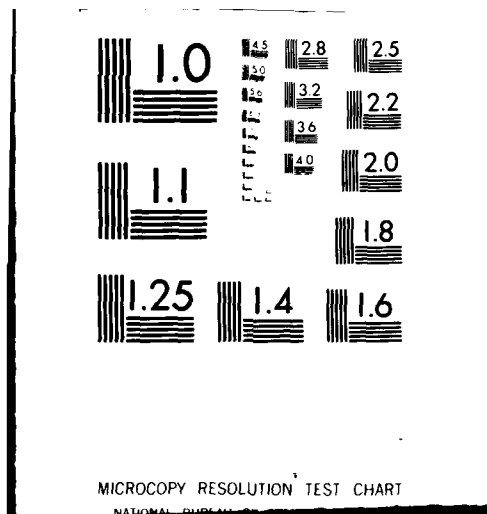
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# INFINITE DIMENSIONAL DYNAMICAL SYSTEMS

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Jack K. Hale

## ABSTRACT

An approach is outlined for the discussion of the qualitative theory of infinite dimensional dynamical systems. Retarded functional differential equations and parabolic partial differential equations are used to illustrate the usefulness of the approach and the limitations of our present knowledge.



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The purpose of this paper is to outline an approach to the development of a theory of dynamic systems in infinite dimensions which is analogous to the theory for finite dimensions. The first problem is to find a class for which there is some hope of classification and yet general enough to include some interesting applications. My goal has been to discover something about a class which includes retarded functional differential equations (RFDE), certain types of neutral functional differential equations (NFDE), parabolic partial differential equations (parabolic PDE) and some other special PDE's. The underlying theory of RFDE's and NFDE's can be found in Hale [7] and parabolic PDE's in Henry [14]. For some details of how these equations fit into the abstract framework below, see Hale [8].

Let  $X, Y, Z$  be Banach spaces and let  $\mathcal{Q}^r = C^r(Y, Z)$ ,  $r \geq 1$ , be the set of functions from  $Y$  to  $Z$  which are bounded and uniformly continuous together with their derivatives up through order  $r$ . We impose the usual topology on  $\mathcal{Q}^r$ . (In applications, other topologies may be needed; for example, the Whitney topology.) For each  $f \in \mathcal{Q}^r$ , let  $T_f(t) : X \rightarrow X$ ,  $t \geq 0$ , be a strongly continuous semigroup of transformations on  $X$ . For each  $x \in X$ , we suppose  $T_f(t)x$  is defined for  $t \geq 0$  and is  $C^r$  in  $x$ .

We say a point  $x_0 \in X$  has a backward extension if there is a  $\varphi : (-\infty, 0] \rightarrow X$  such that  $\varphi(0) = x_0$  and  $T_f(t)\varphi(\tau) = \varphi(t+\tau)$  for  $0 \leq t \leq -\tau$ ,  $\tau \leq 0$ . If there is a backward extension  $\varphi$  through  $x_0$ , we define  $T_f(t)x_0 = \varphi(t)$ ,  $t \leq 0$ . A set  $M \subset X$  is invariant if, for each  $x \in M$ ,  $T_f(t)x$  is defined and belongs to  $M$  for  $t \in (-\infty, \infty)$ . The orbit  $\gamma^+(x)$  through  $x$  is defined as  $\gamma^+(x) = \bigcup_{t \geq 0} T_f(t)x$ .



Let

$$A_f = \{x \in X : T_f(t)x \text{ is defined and bounded} \\ \text{for } t \in (-\infty, \infty)\}.$$

The set  $A_f$  contains much of the interesting information about the semigroup  $T_f(t)$ . In fact, it is very easy to verify the following result.

Proposition 1. If  $A_f$  is compact, then  $A_f$  is maximal, compact, invariant. If, in addition, all orbits have compact closure, then  $A_f$  is a global attractor. Finally, if  $T_f(t)$  is one-to-one on  $A_f$ , then  $T_f(t)$  is a continuous group on  $A_f$ .

The first difficulty in infinite dimensional systems is to decide how to compare two semigroups  $T_f(t)$ ,  $T_g(t)$ . It seems to be almost impossible to make a comparison of any systems on all or even an arbitrary bounded set of  $X$ . If  $A_f$  is compact, Proposition 1 indicates that all essential information is contained in  $A_f$ . Thus, we define equivalence relative to  $A_f$ .

Definition 2. We say  $f$  is equivalent to  $g$ ,  $f \sim g$ , if there is a homeomorphism  $h: A_f \rightarrow A_g$  which preserves orbits and the sense of direction in time. We say  $f$  is structurally stable if there is a neighborhood  $V$  of  $f$  in  $\mathcal{X}^r$  such that  $g \in V$  implies  $g \sim f$ . We say  $f$  is a bifurcation point if  $f$  is not structurally stable.

The basic problem is to discuss detailed properties of the set  $A_f$  and to determine how  $A_f$  and the structure of the flow on  $A_f$  change with  $f$ .

If  $A_f$  is not compact, very little is known at this time. It becomes important therefore to isolate a class of semigroups for which  $A_f$  is compact.

If  $T_f(t)$  is an  $\alpha$ -contraction for  $t > 0$  and  $T_f(t)$  is compact dissipative, then it was proved by Hale and Lopes [11] (see, also, Hale [9], Massatt [27,28]) that  $A_f$  is compact, uniformly asymptotically stable and attracts bounded sets if orbits of bounded sets are bounded. We do not define an  $\alpha$ -contraction, but a special case which is very important in the applications is

$$T_f(t) = S_f(t) + U_f(t), \quad t \geq 0,$$

where  $S_f(t)$  is a strict contraction for  $t > 0$  and  $U_f(t)$  is completely continuous for  $t \geq 0$ . Compact dissipative means there is a bounded set  $B$  in  $X$  such that for any compact set  $K$  in  $X$ , there is a  $t_0 = t_0(K, B)$  such that  $T_f(t) K \subset B$ ,  $t \geq t_0$ .

If  $T_f(t)$  is completely continuous for  $t \geq r$  for some  $r > 0$ , then it was shown by Billotti and LaSalle [1] that  $A_f$  is compact if  $T_f(t)$  is point dissipative. Other conditions for  $A_f$  to be compact which are very useful in the applications have been given recently by Massatt [27,28].

Before proceeding further, we give two examples of semigroups which can be used as models to illustrate several of the ideas.

Suppose  $u \in \mathbb{R}^k$ ,  $x \in \mathbb{R}^n$ ,  $\Omega$  is a bounded, open set in  $\mathbb{R}^n$  with smooth boundary,  $D$  is a  $k \times k$  constant diagonal, positive matrix,  $\Delta$  is the Laplacian operator, and consider the equation

$$u_t - D\Delta u = f(x, u, \text{grad } u) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega.$$

Other boundary conditions could also be used. Let  $W = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$  be

the domain of  $-\Delta$  and let  $X = W^\alpha$ ,  $0 \leq \alpha \leq 1$ , be the domain of the fractional power  $(-\Delta)^\alpha$  of  $-\Delta$  with the graph norm. Under appropriate conditions on  $f, u$ , this equation generates a strongly continuous semigroup  $T_f(t)$  on  $X$  which is compact for  $t > 0$ . In this case  $\mathcal{Q}^r = C^r(\Omega \times \mathbb{R}^k \times \mathbb{R}^{kn}, \mathbb{R}^k)$ . If  $f$  is independent of  $x$ , then  $\mathcal{Q}^r = C^r(\mathbb{R}^k \times \mathbb{R}^{kn}, \mathbb{R}^k)$ . If  $f$  depends only on  $u$ , then  $\mathcal{Q}^r = C^r(\mathbb{R}^k, \mathbb{R}^k)$ . In each of these cases, the theory will be different.

As another example, suppose  $r > 0$ ,  $C = C([-r, 0], \mathbb{R}^n)$ ,  $\mathcal{Q}^r = C^r(C, \mathbb{R}^n)$ ,  $r \geq 1$ , and consider the RFDE,

$$\dot{x}(t) = f(x_t)$$

where, for each fixed  $t$ ,  $x_t$  designates the restriction of a function  $x$  as  $x_t(\theta) = x(t+\theta)$ ,  $-r \leq \theta \leq 0$ . For any  $\varphi \in C$ , let  $x(\varphi)(t)$ ,  $t \geq 0$ , designate the solution with  $x_0(\varphi) = \varphi$  and define  $T_f(t)\varphi = x_t(\varphi)$ . If this function is defined for  $t \geq 0$ , then  $T_f(t) : C \rightarrow C$  is a strongly continuous semigroup and  $T_f(t)$  is completely continuous for  $t \geq r$  if it takes bounded sets to bounded sets.

For differential difference equations

$$\dot{x}(t) = f(x(t), x(t-r))$$

$$\dot{x}(t) = f(x(t-r))$$

the space  $\mathcal{Q}^r$  is respectively,  $C^r(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ ,  $C^r(\mathbb{R}^n, \mathbb{R}^n)$ .

We now begin our discussion of the set  $A_f$ . There are few general results which are independent of  $f$ . However, there is an important one concerning the dimension.

Theorem 3. (Mallet-Paret, Mañé) If  $A_f$  is compact and  $D_x T_f(t)x$  is the sum of a contraction and a completely continuous operator, then  $A_f$  has finite limit capacity. In particular,  $A_f$  has finite Hausdorff dimension.

Mallet-Paret [21] proved the part on finite Hausdorff dimension for the case in which  $X$  is a Hilbert space. Mañé [24] proved the general case, but a different type of analysis was required.

This result has some important implications if one uses the results of Cartwright [3,4] on almost periodic functions. In particular, one can prove the following result.

Corollary 4. If  $A_f$  is compact and the hypothesis of Theorem 3 is satisfied, then there is an integer  $N = N_f$  such that, if  $T_f(t)x$  is almost periodic in  $t$ , then  $T_f(t)x$  is quasiperiodic with frequency base of dimension  $\leq N$ .

Landau and Lifschitz have proposed a principle for the onset of turbulence which consists in the successive introduction through bifurcation of independent frequencies in the oscillatory motion. If the motion is known to be described by the Navier-Stokes equations, then the results of Ladyzenskaya [17] show that  $A_f$  is compact and the hypothesis of Theorem 3 is satisfied. Thus, the Landau-Lifschitz principle cannot be valid as a consequence of Corollary 4.

Other than Theorem 3, there are no general results on  $A_f$ . The research has followed along the lines of considering special types of equations which lead to the semigroup  $T_f(t)$ ; in particular, functional differential equations and parabolic partial differential equations. On the other hand, to explain some of the results that are known, it is convenient to pose general questions.

Q.1. Is  $T_f(t)$  one-to-one on  $A_f$  generically in  $f$ ?

- Q.2. If  $f$  is structurally stable, is  $T_f(t)$  one-to-one on  $A_f$ ?
- Q.3. When is  $A_f$  a manifold or a finite union of manifolds?
- Q.4. Can  $A_f$  be embedded in a finite dimensional manifold generically in  $f$ ?
- Q.5. For each  $x \in A_f$ , is  $T_f(t)x$  continuously differentiable in  $t \in \mathbb{R}$ ?
- Q.6. Are Kupka-Smale semigroups generic?
- Q.7. Are Morse-Smale systems open and structurally stable?

Before discussing specific results, one important observation must be made. All of the above questions are posed for  $A_f$ . This set is much smaller than  $X$  and, thus, the questions have a better chance of being answered. Also, Q.5 is not even meaningful on the whole space for several important applications.

For Q.1, one-to-oneness of  $T_f(t)$  on  $A_f$ , there is no general result known. However, for retarded functional differential equations and special types of neutral functional differential equations, it follows from Nussbaum [33], Hale [7] that  $T_f(t)$  is one-to-one on  $A_f$  if  $f$  is analytic. This is proved by showing that  $T_f(t)x$  is an analytic function of  $t$  for each  $x \in A_f$ . Since  $(T_f(t)x)(\theta) = (T_f(t+\theta)x)(0)$  for all  $\theta$ , this implies  $x \in C([-r,0], \mathbb{R}^n)$  is also analytic. It is easy to construct examples where  $f$  is analytic in a retarded functional differential equation and  $T_f(t)$  is not one-to-one on  $C$ . A trivial example is  $\dot{x}(t) = 0$ ,  $t \geq 0$ ,  $x(t) = \varphi(t)$ ,  $t \in [-1,0]$ ,  $\varphi \in C$ . Non-trivial examples may be found in Hale [7].

For some types of parabolic equations, the results of Henry [14], Miller [32]

(see Manselli and Miller [25] for further references) imply that  $T_f(t)$  is one-to-one on all of  $X$ . For these equations, it would be interesting to study more detailed properties of the solutions on  $A_f$ . For example, if  $f$  is analytic, when are the solutions of  $u_t = \Delta u + f(u)$  in a bounded domain, with some boundary conditions, analytic in  $t$  and the space variable? A personal communication from D. Henry for one space variable shows that, in one space dimension, this analyticity holds for all solutions on the unstable manifold of a hyperbolic equilibrium point. The same conclusion is probably true for  $A_f$ .

In a personal communication, J. Mallet-Paret has given an example where  $T_f(t)$  is not one-to-one on  $A_f$ . However, it is not structurally stable and, thus, the question Q.2 is posed.

For retarded functional differential equations defined on a compact manifold  $M$  without boundary, which are close in some sense to an ordinary differential equation (for example, a differential difference equation with one delay which is small), Kurzweil [16] has shown that  $A_f$  is diffeomorphic to  $M$ . Oliva [35] has generalized these results giving other conditions which imply  $A_f$  is diffeomorphic to  $M$ . The corresponding problems for parabolic equations have not been discussed. However, there should be some analogue.

For certain gradient systems of parabolic equations, Henry [14,15] has shown that  $A_f$  is the union of a finite number of manifolds.

If it is known that the number of critical points is finite and the  $\omega$ -limit set of every orbit is a hyperbolic equilibrium point, then  $A_f$  will be

the union of a finite number of manifolds; namely,  $A_f = \bigcup_i W_f^u(\alpha_i)$  where  $W_f^u(\alpha_i)$  is the unstable manifold (necessarily finite dimensional) of the equilibrium point  $\alpha_i$ .

For Q.5, the differentiability of  $T_f(t)x$  in  $t$  for  $x \in A_f$  is known for some special cases. For RFDE's, this is obviously true since  $T_f(t)x$  is defined for  $t \leq 0$ . We remark that this is true for  $x \in \Lambda_f$  and not for every  $x \in C$ . For certain NFDE's, it is also known to be true (see Hale [7]).

The results in PDE's generally relate to the differentiability of  $T_f(t)x$  for  $x$  in a very large class (see Marsden and McCracken [26]). It should be possible to obtain better results if one restricts  $x$  to be in  $\Lambda_f$ .

In a personal communication, O. Lopes has shown that periodic orbits are always continuously differentiable for the abstract semigroups  $T_f(t) = S_f(t) + U_f(t)$  above.

In studying the properties of semigroups  $T_f(t)$  which are generic in  $f$ , the "size" of the space of functions  $f$  plays an important role. For example, if one is attempting to prove that a periodic orbit may be assumed to be hyperbolic generically in  $f$ , then the space of functions must be large enough to move the characteristic multipliers that are on the unit circle in any direction whatsoever by an appropriate variation of  $f$ . The same difficulty arises with any other property being discussed. When there are more restrictions on the vector field, the characterization of properties which are generic becomes more difficult. We are certainly familiar with similar difficulties in finite dimensional problems; for example, restrictions to vector fields corresponding to Hamiltonian systems, electric circuits, learning models, population

models, etc. In infinite dimensional systems, there is even more flexibility in the choice of the vector field. These restrictions sometimes may be natural or may be imposed to make the problem easier to discuss. Also, each system has a specific role to play in applications.

For retarded functional differential equations, for example, one could be considering either of the following equations

$$\dot{x}(t) = f(x_t)$$

$$\dot{x}(t) = f(x(t), x(t-1))$$

$$\dot{x}(t) = f(x(t-1))$$

with, respectively,  $f \in C^r([-1,0], \mathbb{R}^n)$ ,  $f \in C^r(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ ,  $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$ . All of these equations are retarded functional differential equations  $\dot{x}(t) = F(x_t)$  with  $F$ , respectively, being  $F(\varphi) = f(\varphi)$ ,  $F(\varphi) = f(\varphi(0), \varphi(-1))$ ,  $F(\varphi) = f(\varphi(-1))$ . For the first two cases, Mallet-Paret [21,22] has shown that the Kupka-Smale systems are generic.

For the third case,  $\dot{x}(t) = f(x(t-1))$ , nothing is known except that one may assume the equilibrium points are hyperbolic generically. This is especially interesting since this equation is certainly the one that is most often discussed in the literature as far as the existence of periodic orbits is concerned (see Nussbaum [33] for references). On the other hand, in many of the applications, this equation arose through a transformation of variables of an equation of the form

$$\dot{y}(t) = ay(t) + g(y(t-1)).$$

Perhaps one can prove that the Kupka-Smale systems are generic in this class.



For parabolic equations, very little is known about Kupka-Smale systems. There are even several technical difficulties that arise in the discussion of hyperbolicity of the equilibrium solutions. To be more specific, consider the scalar, one-dimensional parabolic equation

$$\begin{aligned} u_t &= u_{xx} + g(x,u), & 0 < x < 1 \\ u &= 0 & \text{at } x = 0, x = 1, \end{aligned}$$

where  $g \in C^r(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ . In this case, it is not difficult to show that the equilibrium points are hyperbolic generically in  $g$ . The reason for this is the fact that the function  $g$  is allowed to depend upon  $x$ .

On the other hand, if  $g(x,u) = f(u)$  where  $f \in C^r(\mathbb{R}, \mathbb{R})$  is independent of  $x$  and  $u^0(x)$  is an equilibrium point for  $f^0$ , then the linear variational equation about  $u^0$  depends on  $x$ . One must now prove that it is possible to move the spectrum of this linear operator by choosing  $f(u)$  in a neighborhood of  $f^0(u)$  independent of  $x$ . This is a nontrivial problem. Smoller and Wasserman [37] have given an example where hyperbolicity can be determined in a class of  $f(u)$ . Brunovsky and Chow [2] have shown that it always occurs generically in  $f$ . More precisely, suppose  $u(x,n,f)$  is the solution of  $u_{xx} + f(u) = 0$  with  $u(0) = 0$ ,  $u_x(0) = n$  and let  $D_f = \{n : u(x,n,f) \text{ is zero for some } x > 0\}$ . For any  $n \in D_f$ , let  $T(n,f) > 0$  be the first positive zero of  $u(x,n,f)$ . Brunovsky and Chow [2] prove the following result.

Theorem 4. There is a residual set  $\mathcal{S} \subset \{\text{space of } C^2 \text{ functions } f \text{ endowed with the Whitney topology}\}$  such that, for any  $f \in \mathcal{S}$ , the function  $T(n,f)$  is a Morse function. Furthermore, there is a residual set  $\mathcal{S}' \subset \mathcal{S}$  such that, for each  $f \in \mathcal{S}'$ , if  $T(n,f) = 1/n$  for some integer  $n = 1, 2, \dots$ , then

$\partial T(n, f) / \partial n \neq 0$ . In particular, the equilibrium points of

$$\begin{aligned} u_t &= u_{xx} + f(u) & 0 < x < 1 \\ u &= 0 & \text{at } x = 0, 1 \end{aligned} \tag{1}$$

are hyperbolic generically in  $f$ .

This theorem also implies that, generically in  $f$ , the bifurcation of equilibrium points occur as saddle-node bifurcations for Eq. (1)

It does not seem to be possible to extend the proof of Brunovsky and Chow [2] to several space dimensions. Also, in several space dimensions, one probably should include the domain  $\Omega$  with the vector field  $f$  in the discussion of generic properties since the shape of  $\Omega$  sometimes determines the multiplicity of eigenvalues.

The discussion of the genericity of the property of the transversal intersection of stable and unstable manifolds for general parabolic equations has not been considered. Henry [14] has given specific examples where he has shown that this property does hold. These examples are also structurally stable.

For the above scalar one-dimensional parabolic equations on a bounded interval, no one has given an example where the equilibrium points are hyperbolic and the stable and unstable manifolds intersect nontransversally. After spending some time trying, unsuccessfully, to construct such an example, I conjecture that no such example exists. If this is the case, then these equations are Kupka-Smale if and only if the equilibrium points are hyperbolic. Furthermore, the only way the topological structure of the flow can change on a generic set of  $f$  is through saddle-node bifurcations from the result of Brunovsky and

Chow. Even if the above conjecture is generally false, it would be interesting to characterize those  $f$  for which it is true.

A vector field  $f$  is Morse-Smale if it is Kupka-Smale with a finite number of critical points and periodic orbits with the set of nonwandering points  $\Omega(f)$  equal to the set of critical points and periodic orbits. In this case,  $A_f$  should be the union of the unstable manifolds for the critical points and periodic orbits. It is not known if such a system is structurally stable.

One can define Axiom A as in finite dimensions and it would be very interesting to prove that the analogue of the Smale decomposition theorem holds.

Let us now give some simple examples to illustrate several of the remarks above. Consider the scalar one-dimensional parabolic equation

$$\begin{aligned} u_t &= u_{xx} + \lambda(u-u^3), & 0 < x < \pi, \quad t \geq 0 \\ u &= 0 \quad \text{at} \quad x = 0, \pi \end{aligned} \quad (2)$$

where  $\lambda$  is a real parameter. This equation defines a strongly continuous semigroup  $T_\lambda(t)$ ,  $t \geq 0$ , on  $H_0^1(0, \pi)$  (see, for example, Henry [14]).

If

$$V(\varphi) = \int_0^\pi \left[ \frac{1}{2} \varphi_x^2 - \lambda \left( \frac{\varphi^2}{2} - \frac{\varphi^4}{4} \right) \right] dx$$

for  $\varphi \in H_0^1(0, \pi)$ , then  $\dot{V}(u(t, \cdot))$ , the derivative of  $V$  along the solutions of (2), satisfies

$$\dot{V}(u(t, \cdot)) = - \int_0^\pi u_t^2 dx \leq 0.$$

This implies every solution of (2) is bounded for  $t \geq 0$ . Also, every bounded orbit has compact closure which implies the  $\omega$ -limit set exists. By the

invariance principle, the  $\omega$ -limit set of each orbit belongs to the set of equilibrium points; that is, a solution in  $H_0^1(0, \pi)$  of the equation

$$\begin{aligned} 0 &= u_{xx} + \lambda(u - u^3), & 0 < x < \pi, \\ u &= 0 \text{ at } x = 0, \pi. \end{aligned} \quad (3)$$

This result was essentially proved by Chafee and Infante [5]. It is also known that the  $\omega$ -limit set of each bounded solution (even for general vector field  $f(u)$ ) consists of only one equilibrium point (see Matano [30], Hale and Massatt [12]).

The next step is to analyze how  $A_\lambda$ , the maximal compact invariant set, depends upon  $\lambda$ . The eigenvalues of the linear variational equation for the zero solution are  $\lambda_n = n^2$ ,  $n = 1, 2, \dots$ . At each  $\lambda_n$ , two hyperbolic solutions  $\varphi_n^+, \varphi_n^-$  bifurcate from zero. Henry [14] proves that these are the only points of bifurcation and, for  $\lambda_n < \lambda < \lambda_{n+1}$ , there are exactly  $2n+1$  equilibrium solutions with the unstable manifolds  $W^u(\varphi_j^+), W^u(\varphi_j^-)$  of  $\varphi_j^+, \varphi_j^-$  having dimension  $j - 1$ . In this interval of  $\lambda$ , the unstable manifold  $W^u(0)$  of 0 has dimension  $n$ .

Let  $E_\lambda$  be the set of equilibrium points for a given  $\lambda$ . For  $\lambda_n < \lambda < \lambda_{n+1}$ , the set  $A_\lambda$  is given by  $A_\lambda = \bigcup_{\varphi \in E_\lambda} W^u(\varphi) = \text{closure of } W^u(0)$  and has dimension  $n$ . If the stable and unstable manifolds of the equilibrium points always intersect transversally, then Eq. (2) is structurally stable. Henry proves this is the case for  $0 < \lambda < 1$ ,  $1 < \lambda < 4$ ,  $4 < \lambda < 9$ ,  $9 < \lambda < 16$ . For the latter interval, the oddness of  $f(u) = u - u^3$  was exploited. The other cases hold for general  $f(u)$ .

It is the belief of the author that the transversal intersection property

always holds and the proof of this fact must exploit more detailed properties of the solutions of Eq. (2). The recent results of Matano [31] on the change of complexity of solutions with increasing time perhaps could play a role and do, in fact, make it easier to obtain the results of Henry [14]. It is reasonable to refer to Eq. (2) as a gradient flow since formally  $\dot{u}_t = -\text{grad } V(u)$  in  $H_0^1(0, \pi)$ . One can define gradient flows in several space dimensions and vector functions  $u$  (see Henry [14]). It would be very interesting to know if gradient Morse-Smale systems are open and dense. No results in this direction seem to be available.

The scalar one-dimensional equation (2) behaves qualitatively as a scalar ordinary differential equation with the analogy being complete if we knew that stable and unstable manifolds for hyperbolic equilibria always intersect transversally. In several respects, retarded functional differential equations generate semigroups which have several properties in common with parabolic equations. However, in detail, these equations behave quite differently with the retarded equations having a more complicated orbit structure. We give an example to illustrate these remarks.

Consider the scalar equation

$$\dot{x}(t) = - \int_{-1}^0 a(-\theta) g(x(t+\theta)) d\theta \quad (4)$$

where  $g \in C^2(\mathbb{R}, \mathbb{R})$ ,  $a \in C^2([0, 1], \mathbb{R})$ ,  $G(x) = \int_0^x g \rightarrow \infty$  as  $|x| \rightarrow \infty$ ,  $a(1) = 0$ ,  $a(s) \geq 0$ ,  $\dot{a}(s) \leq 0$ ,  $\ddot{a}(s) \geq 0$ . We consider Eq. (3) for initial data in  $C([-1, 0], \mathbb{R})$ .

For Eq. (4), Levin and Nohel [20] exhibited a Liapunov function

$$V(\varphi) = G(\varphi(0)) - \frac{1}{2} \int_{-r}^0 \dot{a}(-\theta) \left[ \int_{\theta}^0 g(\varphi(s)) ds \right]^2 d\theta$$

whose derivative  $\dot{V}$  along the solutions of (4) is given by

$$\dot{V}(\varphi) = \frac{1}{2}\dot{a}(r)\left[\int_{-r}^0 g(\varphi(\theta))d\theta\right]^2 - \frac{1}{2}\int_{-r}^0 \ddot{a}(-\theta)\left[\int_{\theta}^0 g(\varphi(s))ds\right]^2 d\theta \leq 0.$$

Using the invariance principle, one can then prove the following result (see Hale [7]).

Theorem 5. Every orbit of Eq. (4) is bounded and has an  $\omega$ -limit set. If  $g$  has isolated zeros, then

- (i) If there is an  $s$  such that  $\ddot{a}(s) > 0$ , then the  $\omega$ -limit set of any orbit is a constant function corresponding to a zero of  $g$ ;
- (ii) If  $\ddot{a}(s) = 0$  for all  $s$  and  $a \neq 0$ , then the  $\omega$ -limit set of any orbit is a periodic orbit of period 1 generated by a 1-periodic solution of the ordinary equation

$$\ddot{x} + a(0)g(x) = 0.$$

Let us consider first a special case of (i); namely,  $\ddot{a}(s) > 0$  for  $s \in (0,1)$ , and the zeros  $\alpha_1 < \alpha_2 < \dots < \alpha_{2k+1}$  of  $g$  are simple. Let  $A_{a,g}$  be the attractor for Eq. (4). We want to study the dependence of  $A_{a,g}$  on  $g$ , keeping  $a$  fixed. One can show that each equilibrium point is hyperbolic,  $\alpha_{2j+1}$  are stable,  $j = 0,1,\dots,k$ ,  $\alpha_{2j}$  are saddle points with unstable manifold  $W^u(\alpha_{2j})$  having dimension one,  $j = 1,2,\dots,k$ . Thus, the attractor  $A_{a,g} = \bigcup_{s=1}^{2k+1} W^u(\alpha_s)$  is one dimensional.

The basic question is the following: to which equilibrium points do the orbits on an unstable manifold  $W^u(\alpha_{2j})$  tend? Is it always true that these orbits tend to  $\alpha_{2j-1}$  and  $\alpha_{2j+1}$ ? Surprisingly, the answer to the latter

question is negative and there can exist saddle-connections. This means the topological structure of the flow can change without having a bifurcation of equilibrium in contrast to what we believe is happening in the scalar one-dimensional parabolic Eq. (1). To state a precise result, suppose  $k = 2$ , that is, there are five simple zeros of  $g$ ,  $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$ . If the symbol  $\alpha_{2j}(\alpha_k, \alpha_\ell)$  designates that the saddle point  $\alpha_{2j}$  is connected by its unstable manifold to  $\alpha_k, \alpha_\ell$ , then the orbit structure on  $A_{a,g}$  is determined by a pair  $[\alpha_{2j}(\alpha_k, \alpha_\ell), \alpha_{4i}(\alpha_m, \alpha_n)]$ . Hale and Rybakowski [15] have shown that each of the following orbit structures can be attained by choosing  $g$  appropriately in the above class:

$$\{\alpha_2(\alpha_1, \alpha_3), \alpha_4(\alpha_3, \alpha_5)\}$$

$$[\alpha_2(\alpha_1, \alpha_4), \alpha_4(\alpha_3, \alpha_5)]$$

$$[\alpha_2(\alpha_1, \alpha_5), \alpha_4(\alpha_3, \alpha_5)]$$

$$[\alpha_2(\alpha_1, \alpha_5), \alpha_4(\alpha_2, \alpha_5)]$$

$$[\alpha_2(\alpha_1, \alpha_3), \alpha_4(\alpha_1, \alpha_5)].$$

The first case corresponds to the natural order of the reals on  $A_{a,g}$ ; the second and fourth cases correspond to saddle connections; the third and fifth cases reverse the natural order of the reals. The first, third and fifth cases remain for small perturbations of  $g$ . Although the second and fourth cases seem as if they should not remain after appropriately small perturbations of  $g$ , but the authors have been unable to prove this fact and thus, the question is open: Is the set of  $g$  for which saddle-connections exist nongeneric?

Let us now suppose that  $a(s) = a_0(s)$  is linear and, in particular,

that  $a_0(s) = 4\pi^2(1-s)$ . Also, suppose  $g$  is restricted to the class of functions such that  $xg(x) > 0$  for  $x \neq 0$ ,  $g'(0) = 1$ . Then the linear variational equation about  $x = 0$  has two eigenvalues on the imaginary axis with the remaining ones having negative real parts. In this case, it is natural to discuss the bifurcation of periodic orbits from zero which arise by small variations in  $a_0$  or  $g$ . To do this, one must compute the bifurcation function at  $(a_0, g)$  and determine the first nonvanishing coefficients in the Taylor expansion. The generic Hopf bifurcation corresponds to the coefficient  $\alpha_{a_0, g}$  of the cubic terms being  $\neq 0$ . It has been shown by Hale [10] that  $\alpha_{a_0, g} = 0$  for all  $g$  in the above class and, therefore, a generic Hopf bifurcation can never occur. It is not known if there ever exist any coefficient in the Taylor series of the bifurcation function which is not zero.

The two preceding examples illustrate clearly that important questions in the qualitative theory of infinite dimensional systems arise in the simplest of systems. It is not easy to outline a general direction of research that should be pursued. On the other hand, it is clear that one should first determine the extent to which infinite dimensional systems share the general qualitative properties that are known for ordinary differential equations; for example, the genericity of Kupka-Smale systems, the structural stability properties of Morse-Smale systems, the decomposition theorem of Smale for systems that satisfy Axiom A, etc.

The preceding discussion shows that these questions are difficult in the most elementary examples. Examples of this type need to be discussed in great detail in order to develop the methods and intuition to proceed to more



general cases. In scalar one-dimensional parabolic equations above, for instance, important properties of the system were not being used extensively - in particular, the maximum principle. What role, if any, does it play in these qualitative investigations?

The example of the retarded equation exhibited flows with a more complicated qualitative behavior by the introduction of saddle connections. Perhaps this is to be expected since the equation must define a flow in infinite dimensions. On the other hand, so did the parabolic equation. What makes the retarded equation more complicated? What types of flows can be generated by a scalar retarded equation?

Very complicated chaotic type of motions have been observed in applications of scalar retarded equations (see, for example, Lasota and Watewska-Czyrwska [19], Lasota [18], Glass and Mackey [6]). Peters [56], Walther [38] have actually proved that chaos occurs in equations  $\dot{x}(t) = f(x(t-1))$  for some nonlinear functions  $f$ . It should not be surprising that complicated dynamics might occur since mappings on the interval exhibit such behavior. However, this is pure speculation and the process involved is not understood. For example, consider the equation

$$\dot{x}(t) = x(t) - f_{\lambda}(x(t-1)) \quad (5)$$

where  $f_{\lambda}(x)$  depends on a real parameter  $\lambda$ . Suppose that the interval map  $x(t) = f_{\lambda}(x(t-1))$  has successive bifurcations of periodic points through period doubling as  $\lambda$  increases and, for some value of  $\lambda$ , "chaotic" motion occurs. Under what conditions do these periodic points correspond to periodic orbits of (5) and when does a chaotic motion occur in (5)? A version of this

problem presently is being studied by Chow and Mallet-Paret for the equation

$$\mu \dot{x}(t) = x(t) - f_{\lambda}(x(t-1)) \quad (6)$$

where  $\mu$  is a small parameter. Preliminary investigations indicate that the bifurcation phenomena in (6) is much more complicated than the corresponding one for the interval map.

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